B.sc(H) part 1 paper 1
Topic:Resolution into factors
subject:mathematics
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2. Resolution of cos θ into factors

Proceeding as in the previous art, we get

$$\sin\theta = 2^{p-1}\sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\cdot\sin\frac{2\pi+\theta}{p}\sin\frac{(p-1)\pi+\theta}{p}$$

where $p = 2^n$.

Changing θ into $\frac{\pi}{2}+\theta$ in the above equation, we get

$$\cos\theta = 2^{p-1}\sin\frac{\pi+2\theta}{2p}\cdot\sin\frac{3\pi+2\theta}{2p}\ldots\sin\frac{(2p-1)\pi+2\theta}{2p}$$

The last factor $= \sin\left\{\pi-\frac{\pi-2\theta}{2p}\right\} = \sin\frac{\pi-2\theta}{2p}$.

The last factor but one $= \sin\frac{(2p-3)\pi+2\theta}{2p} = \sin\frac{3\pi-2\theta}{2p}$ and so on.

Now combining together the first and the last factor, the second and the last but one and so on, and keeping in view the argument in the previous article.

$$\cos\theta = 2^{p-1}\left\{\sin\frac{\pi+2\theta}{2p}\cdot\sin\frac{\pi-2\theta}{2p}\right\}\left\{\sin\frac{3\pi+2\theta}{2p}\cdot\sin\frac{3\pi-2\theta}{2p}\right\}\ldots\left\{\sin\frac{(p-1)\pi+2\theta}{2p}\cdot\sin\frac{(p-1)\pi-2\theta}{2p}\right\}$$

$$\Rightarrow \cos\theta = 2^{p-1}\left\{\sin^2\frac{\pi}{2p}-\sin^2\frac{2\theta}{2p}\right\}\left\{\sin^2\frac{3\pi}{2p}-\sin^2\frac{2\theta}{2p}\right\}\ldots\left\{\sin^2\frac{(p-1)\pi}{2p}-\sin^2\frac{2\theta}{2p}\right\} \quad \ldots(1)$$

Let θ → 0, therefore

$$1 = 2^{p-1}\sin^2\frac{\pi}{2p}\cdot\sin^2\frac{3\pi}{2p}\ldots\sin^2\frac{(p-1)\pi}{2p} \quad \ldots(2)$$

Dividing (1) by (2)

$$\cos\theta = \left\{1 - \frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{\pi}{2p}}\right\}\left\{1 - \frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{3\pi}{2p}}\right\}\cdots\left\{1 - \frac{\sin^2\frac{2\theta}{2p}}{\sin^2\frac{(p-1)\pi}{2p}}\right\} \quad \ldots(3)$$

Thus for all values of θ under consideration

$$\cos\theta = \prod_{r=1}^{p/2}\left\{\frac{1-\sin^2\frac{2\theta}{2p}}{\sin^2\frac{(2r-1)\pi}{2p}}\right\}$$

Now let $p \to \infty$, therefore

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\cdots$$

$$= \prod_{r=1}^{\infty}\left\{1 - \frac{4\theta^2}{(2r-1)^2\pi^2}\right\}$$

.3 To deduce the expansion of cos θ as an infinite product by assuming the expansion of sin θ as an infinite product

We have, $\sin 2\theta = 2 \sin\theta \cdot \cos\theta$

$$\therefore \quad \cos\theta = \frac{\sin 2\theta}{2\sin\theta} = \frac{(2\theta)\prod_{r=1}^{\infty}\left\{1-\frac{(2\theta)^2}{r^2\pi^2}\right\}}{2(\theta)\prod_{r=1}^{\infty}\left\{1-\frac{\theta^2}{r^2\pi^2}\right\}}$$

$$= \frac{\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{2^2\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{4^2\pi^2}\right)\cdots}{\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{4^2\pi^2}\right)\cdots}$$

Now, the factors in the numerator for which r is even cancel with those in the denominator. Of course, such a cancellation is valid, for the products involved are convergent. Therefore

$$\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots$$

.4 Sums of powers of the reciprocals of all natural numbers

We have, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

and also $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots$

Thus, $\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots = \theta\left[1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\cdots\right]$

$$\Rightarrow \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots = \left(1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\cdots\right)$$

Taking the logarithms of both sides, we have

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\ldots+\log\left(1-\frac{\theta^2}{r^2\pi^2}\right)+\ldots$$

$$= \log\left\{1-\left(\frac{\theta^2}{3!}-\frac{\theta^4}{5!}+\ldots\right)\right\}$$

$$\Rightarrow \sum_{r=1}^{\infty}\log\left(1-\frac{\theta^2}{r^2\pi^2}\right) = \log(1-y) \text{ where } y = \frac{\theta^2}{3!}-\frac{\theta^4}{5!}+\ldots$$

Expanding each of the logarithms in the above identity, we have

$$\sum_{1}^{\infty}\left(\frac{\theta^2}{r^2\pi^2}+\frac{\theta^4}{2r^4\pi^4}+\frac{1}{3}\cdot\frac{\theta^6}{r^6\pi^6}+\ldots\right) = y+\frac{1}{2}y^2+\frac{1}{3}y^3+\ldots$$

$$\Rightarrow \theta^2\sum_{1}^{\infty}\frac{1}{r^2\pi^2}+\theta^4\sum_{1}^{\infty}\frac{1}{2r^4\pi^4}+\theta^6\sum_{1}^{\infty}\frac{1}{3r^6\pi^6}+\ldots$$

$$= \left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\ldots\right)+\frac{1}{2}\left(\frac{\theta^4}{6}-\frac{\theta^4}{120}+\ldots\right)^2+\ldots \qquad \ldots(1)$$

Since the equation (1) is true for all values of θ, the coefficient of θ^2 on both sides must be the same, and similarly those of θ^4 and so on.

Hence we have, $\sum_{1}^{\infty}\frac{1}{r^2\pi^2} = \frac{1}{6}$

$$\sum\frac{1}{2r^4\pi^4} = \frac{1}{180}, \text{ and so on.}$$

Hence $\quad \sum\frac{1}{r^2} = \frac{\pi^2}{6}$ and $\sum\frac{1}{r^4} = \frac{\pi^4}{90}$ etc.

In other words $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}$

and $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots=\frac{\pi^4}{90}$

By proceeding in a similar manner

$$\prod_{r=1}^{\infty}\left(1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right)=\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\ldots$$

so that $\sum\log\left\{1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}=\log(1-y)$

where $y=\frac{\theta^2}{2!}-\frac{\theta^4}{4!}+\ldots$

Thus, $\sum_1^{\infty}\left(\frac{4\theta^2}{(2r-1)^2\pi^2}+\frac{1}{2}\frac{16\theta^4}{(2r-1)^4\pi^4}+\ldots\right\}=y+\frac{1}{2}y^2+\frac{1}{3}y^3+\ldots$

$$\Rightarrow\quad \theta^2\sum_1^{\infty}\frac{4}{(2r-1)^2\pi^2}+\theta^4\sum_1^{\infty}\frac{8}{(2r-1)^4\pi^4}+\ldots$$

$$=\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\ldots\right)+\frac{1}{2}\left(\frac{\theta^2}{2}-\frac{\theta^4}{24}+\ldots\right)^2+\ldots$$

Hence equating the coefficient of θ^2 and θ^4, we have

$$\sum\frac{4}{(2r-1)^2\pi^2}=\frac{1}{2};$$

$$\sum\frac{8}{(2r-1)^4\pi^4}=\frac{1}{12}\text{ and so on.}$$

Hence $\sum\frac{1}{(2r-1)^2}=\frac{\pi^2}{8}$ and $\sum\frac{1}{(2r-1)^4}=\frac{\pi^4}{96}$ etc.

In other words $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$

and $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$